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THE CUSPS OF ZEEMAN'S CATASTROPHE MACHINE

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W. BURKE asked me the following question. Zeeman's catastrophe machine [1] has four cusps in its bifurcation set; can one construct a catastrophe machine with just two cusps? This note shows that the answer to this question is *no* under rather broad hypotheses. Our answer to the question can be interpreted as a generalization of the four vertex theorem of elementary differential geometry [2]. The proof of our theorem provides yet another proof of the four vertex theorem, our proof being in the spirit of "catastrophe theory." We shall first describe Zeeman's catastrophe machine and state the theorem, then display the connection with differential geometry, and finally prove the theorem.

Briefly recall the construction of Zeeman's catastrophe machine. There is a disk D in the plane which is free to rotate around its center. Two springs S_1 and S_2 are attached to one point p of the boundary of D. The spring S_1 has its other end attached at a fixed point in the plane. The other end v of S_2 is free to be moved in the plane; see Fig. 1.

The quantity which one measures with this machine is the stable equilibrium position(s) of the disk D as a function of the location of $v \in \mathbf{R}^2$. If the position of D is measured by the angle θ which p makes with the x-axis, then one determines the quantity $\theta(v)$. For some values of v, there are two stable equilibria. The curve in $\mathbf{R}^2$ across which the number of stable equilibria changes is called the *bifurcation set* of the machine. The bifurcation set of Zeeman's machine has four cusps; see Fig. 2.

An analysis of the catastrophe phenomenon which occurs across the bifurcation set can be made by introducing a potential function $F(v, \theta)$. The function F measures the potential energy of the system with the disk D held at position θ and the end of the spring S_2 held at v. The stable equilibria of the machine are given by pairs (v, θ) at which F has a local minimum when regarded as a function of θ alone. The bifurcation set consists of $v \in \mathbf{R}^2$ at which the number of local minima change. In particular, a necessary condition for v to lie in the bifurcation set is that there exists a θ for which $(\partial F/\partial\theta)(v, \theta) = (\partial^2 F/\partial\theta^2)(v, \theta) = 0$.*

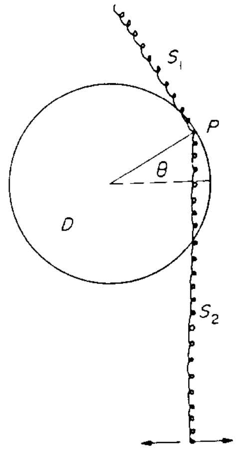

Fig. 1. Zeeman's catastrophe machine.

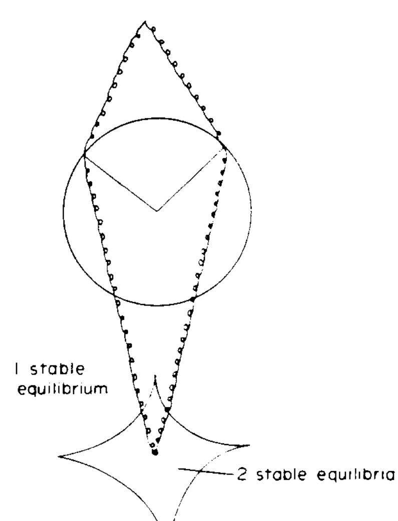

Fig. 2. The bifurcation set of Zeeman's machine.

*The implicit function theorem implies that solutions of the equation $(\partial F/\partial\theta)(v, \theta) = 0$ locally form a manifold described as the graph of a function $\theta(v)$ near points where $(\partial/\partial\theta)(\partial F/\partial\theta) \neq 0$. Since the equation $(\partial F/\partial\theta) = 0$ describes the critical points of F as a function of θ, this justifies the statement in the text.

We shall generalize Zeeman's catastrophe machine to machines describable in terms of potential functions $F: \mathbf{R}^2 \times S^1 \to \mathbf{R}$ (subject to mild non-degeneracy conditions).

THEOREM. *Let $F: \mathbf{R}^2 \times S^1 \to \mathbf{R}$ be a function which satisfies the condition $F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x} \neq 0$. Assume further that there is a smooth map $\phi: S^1 \to \mathbf{R}^2$ so that $F|\phi(\theta) \times S^1$ has a unique, non-degenerate local minimum at θ. Then the bifurcation set of F contains at least four cusps.*

We make three remarks concerning the hypotheses of the theorem. First, both of the examples introduced above do satisfy the hypotheses of the theorem. Second, the condition on the derivatives of F assures that variations of the point $(x, y) \in \mathbf{R}^2$ always produce two linearly independent variations of the potential F affecting the critical points of F. I do not know whether this condition is necessary for the theorem to be true. Third, the hypothesis giving the existence of ϕ corresponds to the fact that the behavior of Zeeman's catastrophe machine is dominated by the spring with variable endpoint when this spring is stretched a lot.

Before proceeding with the development of our theory, let us examine a potential function which establishes the relevance of our theory to the four vertex problem in differential geometry. Let $\phi: S^1 \to \mathbf{R}^2$ be a smooth embedding of S^1 as a closed strictly convex curve parametrized by arc length. Consider the function $F(v, \theta) = \frac{1}{2}(v - \phi(\theta)) \cdot (v - \phi(\theta))$ where $\cdot$ denotes the Euclidean inner product. If we regard $F(v, \theta)$ as a potential function for a catastrophe machine, what can we say about its bifurcation set? Calculate

$$\frac{\partial F}{\partial \theta}(v, \theta) = -\dot{\phi}(\theta) \cdot (v - \phi(\theta)), \tag{1}$$

and

$$\frac{\partial^2 F}{\partial \theta^2}(v, \theta) = \dot{\phi}(\theta) \cdot \dot{\phi}(\theta) - \ddot{\phi}(\theta) \cdot (v - \phi(\theta)). \tag{2}$$

Equation (1) says that the critical points of F as a function of θ occurs at points (v, θ) such that v is on the normal line to the curve ϕ at $\phi(\theta)$. To interpret (2) geometrically, notice that $\dot{\phi}(\theta) \cdot \dot{\phi}(\theta) = 1$ and that $\ddot{\phi}(\theta)$ and $v - \phi(\theta)$ are parallel whenever v is a solution of (1). Thus $|v - \phi(\theta)| = |\ddot{\phi}(\theta)|^{-1}$ for solutions of (1) and (2). Geometrically, v is a center of curvature of ϕ. The bifurcation set of F is contained in the evolute of ϕ, the locus of its centers of curvature. All of the centers of curvature are actually in the bifurcation set in this case because $\partial^2 F/\partial\theta^2$ is a linear function along a normal to ϕ and changes sign at the center of curvature along this normal.

Let us calculate the tangent vector to the bifurcation set in this case. The bifurcation set is explicitly described by the curve $\gamma: S^1 \to \mathbf{R}^2$ with $\gamma(\theta) = \phi(\theta) + [\ddot{\phi}(\theta)/(\ddot{\phi}(\theta) \cdot \ddot{\phi}(\theta))] = \phi(\theta) + (1/K)\nu(\theta)$. Here $\nu(\theta)$ is the unit normal to the image of ϕ at $\phi(\theta)$. Differentiating gives $\dot{\gamma}(\theta) = \dot{\phi}(\theta) - (\dot{K}/K^2)\nu(\theta) + (1/K)\dot{\nu}(\theta)$. Now $\dot{\nu}(\theta) = -K\dot{\phi}(\theta)$, so $\dot{\gamma}(\theta) = (-\dot{K}/K^2)\nu(\theta)$. From this equation, it follows that the bifurcation set is a smooth curve except at points which correspond to critical points for the curvature of ϕ. Thus, *the cusps of the bifurcation set correspond to critical points of the curvature of ϕ*. The four vertex theorem is the statement that the curvature of ϕ has at least four critical points. This is just a special case of the theorem stated above for a particular potential functions F. Before passing on, we calculate the curvature of γ:

$$\ddot{\gamma}(\theta) = -\frac{d}{d\theta}\left(\frac{\dot{K}}{K^2}\right)\nu(\theta) + \frac{\dot{K}}{K}\dot{\phi}(\theta).$$

Hence, the curvature of γ is

$$\frac{\dot{\gamma}(\theta) \times \ddot{\gamma}(\theta)}{|\dot{\gamma}(\theta)|^3} = \left|\frac{K^2}{\dot{K}}\right|^3\left(-\frac{\dot{K}}{K^2}\nu(\theta) \times \frac{\dot{K}}{K}\dot{\phi}(\theta)\right) = \frac{K^3}{|\dot{K}|}.$$

Since ϕ is convex, γ has positive curvature at its regular points.

Return now to the general situation. Assume that $F(v, \theta)$ is the potential for a catastrophe machine. Then the bifurcation set B is contained in the projection of the set $C = \{(v, \theta) | (\partial F/\partial\theta)(v, \theta) = (\partial^2 F/\partial\theta^2)(v, \theta) = 0\}$ onto the v plane. We shall call the set C the *catastrophe set* for F. One expects that C will usually be a smooth curve, locally. Let us compute its tangent line. It is convenient now to use subscripts to denote partial derivatives. The normal to the surface $F_\theta = 0$ is given by

$$\nabla F_\theta = F_{\theta x}\frac{\partial}{\partial x} + F_{\theta y}\frac{\partial}{\partial y} + F_{\theta\theta}\frac{\partial}{\partial \theta}.$$

Similarly, the normal vector to $F_{\theta\theta}=0$ is

$$\nabla F_{\theta\theta} = F_{\theta\theta x}\frac{\partial}{\partial x} + F_{\theta\theta y}\frac{\partial}{\partial y} + F_{\theta\theta\theta}\frac{\partial}{\partial \theta}.$$

The implicit function theorem implies that C is a smooth curve near points where $\nabla F_\theta \times \nabla F_{\theta\theta} \neq 0$. Since $F_{\theta\theta}=0$ along C

$$\nabla F_\theta \times \nabla F_{\theta\theta} = F_{\theta\theta\theta}F_{\theta y}\frac{\partial}{\partial x} - F_{\theta\theta\theta}F_{\theta x}\frac{\partial}{\partial y} + (F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x})\frac{\partial}{\partial \theta} \quad \text{along } C.$$

The hypothesis of the theorem implies that C is a smooth curve. The tangent line to the bifurcation set B is parallel to the vector $F_{\theta y}(\partial/\partial x) - F_{\theta x}(\partial/\partial y)$. We conclude that the bifurcation set is perpendicular to the gradient F_θ with respect to the variables (x, y). Consider next the sets $\Gamma(\theta) = \{(v) \in \mathbf{R}^2 | F(v, \theta)$ has a critical point with respect to θ at $(v, \theta)\}$. The set $\Gamma(\theta_0)$ is the intersection of the surfaces $F_\theta = 0$ and $\theta = \theta_0$. It follows that $\Gamma(\theta)$ will be a smooth curve at all points for which $(F_{\theta y}, -F_{\theta x}) \neq (0,0)$, and this vector will be tangent to $\Gamma(\theta)$. Moreover, the calculation of the last paragraph shows that $\Gamma(\theta)$ meets the bifurcation set B tangentially, provided $F_{\theta\theta\theta} \neq 0$. This discussion is summarized in the following lemma.

LEMMA. *Let $\Gamma(\theta)$ be $\{v \in \mathbf{R}^2 | F(v, \theta)$ has a critical point with respect to θ at $(v, \theta)\}$. Assume that $F_{\theta\theta\theta}(F_{\theta y}, -F_{\theta x}) \neq (0,0)$ at (v, θ) where $v \in \Gamma(\theta) \cap B$. Then $\Gamma(\theta)$ and B are smooth curves in a neighborhood of v which are tangent at v. As θ is varied, the bifurcation set B is described as the envelope of the curves $\Gamma(\theta)$. Additional information is the following.*

LEMMA. *Define $\Gamma_m(\theta)$ to be the subset of $\Gamma(\theta)$ for which F has a local minimum at θ. Let (x, y) be a point of $\Gamma(\theta) \cap B$ such that $F_{\theta\theta\theta} \neq 0$ and $F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x} \neq 0$. Then $\Gamma_m(\theta)$ has an endpoint at (x, y).*

Proof. The directional derivative of $F_{\theta\theta}$ along $\Gamma(\theta)$ is given by $F_{\theta y}F_{\theta\theta x} - F_{\theta x}F_{\theta\theta y} \neq 0$. Therefore, the sign of $F_{\theta\theta}$ changes at (x, y).

Remark. Note that if $\Gamma(\theta)$ is oriented so that (x, y) is the forward endpoint of $\Gamma_m(\theta)$, then the directional derivative is negative. This is an important comment to be used later.

Let us now calculate the curvature of B. The tangent vector to B, parametrized by θ is

$$w(\theta) = \frac{F_{\theta\theta\theta}}{F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x}}(F_{\theta y}, -F_{\theta x}).$$

Denoting $F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x}$ by D, the curvature of B is

$$\frac{w \times \dfrac{dw}{d\theta}}{|w|^3} = \frac{F_{\theta\theta\theta}}{D}\,\frac{(F_{\theta y}, -F_{\theta x}) \times \left[\dfrac{d}{d\theta}\left(\dfrac{F_{\theta\theta\theta}}{D}\right)(F_{\theta y}, -F_{\theta x}) + \dfrac{F_{\theta\theta\theta}}{D}(F_{\theta\theta y}, -F_{\theta\theta x})\right]}{\left|\dfrac{F_{\theta\theta\theta}}{D}\right|^3 (F_{\theta y}^2 + F_{\theta x}^2)^{3/2}}$$

$$= \frac{D|D|}{|F_{\theta\theta\theta}|(F_{\theta y}^2 + F_{\theta x}^2)^{3/2}} = \frac{F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x}}{(F_{\theta y}^2 + F_{\theta x}^2)^{3/2}}\left|\frac{F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x}}{F_{\theta\theta\theta}}\right|.$$

Consider now the situation in which $F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x}$ is never zero and there is a curve $\phi \subset \mathbf{R}^2$ which intersects each $\Gamma_m(\theta)$ just once. Under these assumptions, one has the following lemma.

LEMMA. *Suppose $F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x} \neq 0$ and there is a diffeomorphism $\phi: S^1 \to \mathbf{R}^2$ such that $F(\phi(\theta), \cdot)$ has a unique local minimum at 0. Then each curve $\Gamma_m(\theta)$ has a connected intersection with the interior of ϕ which has one endpoint in B and one endpoint at $\phi(\theta)$.*

Proof. We already know that endpoints of $\Gamma_m(\theta)$ are in B. The additional assertion of the lemma is that $\Gamma_m(\theta)$ has connected intersection with the interior of ϕ. We assert that any component of $\Gamma_m(\theta)$ cannot be contained in the interior of ϕ. The curves $\Gamma(\theta)$ are non-singular curves with tangent lines parallel to $(-F_{\theta y}, F_{\theta x})$. Thus $\Gamma_m(\theta)$ can have an endpoint only at points where $F_{\theta\theta}=0$. But we have seen that $F_{\theta\theta}$ is a monotonic function along each curve. Therefore, no component of $\Gamma_m(\theta)$ in the interior of ϕ can have two endpoints in ϕ. Since the only point of ϕ which can contain a point of $\Gamma_m(\theta)$ is $\phi(\theta)$, the lemma is proved.

We are now able to prove the theorem. Consider the complement of the bifurcation set in the interior of ϕ. The component of this set which contains ϕ in its closure, has at least two components in its boundary. One of the components is ϕ; call another K. The set K is contained in the bifurcation set B; see Fig. 3.

The curve K is a simple closed curve, provided it is not a single point. (This is a degenerate case, in which all of the curves $\Gamma_m(\theta)$ end at a single point). We shall prove that the curve K is concave toward its exterior at all regular points and at points of self intersection. From this the theorem is proved as follows. If we orient K counterclockwise, then the unit tangent vector to K rotates in a clockwise direction at regular points and points of self intersection. On the other hand, the index theorem states that the total change in the angle of the tangent vector around K is $+2\pi$. The only positive contributions come from the cusps of K, and each contribution is $\pm\pi$. Therefore, we conclude that there are at least three cusps in K. Another observation is that the number of cusps in B is even, because they occur at points where $F_{\theta\theta\theta}$ changes sign. Hence, B has at least four cusps, proving the theorem.

Let us now prove that the curve K is concave toward its exterior. The orientation of ϕ determines the orientation of the curves $\Gamma(\theta)$. Using the implicit function theorem to solve $F_\theta(x, y, \theta)$ for θ as a function of (x, y), we calculate that $d\theta = -F_{\theta\theta}^{-1}(F_{\theta x}dx + F_{\theta y}dy)$. If ϕ is orientation preserving, this implies that $-\nabla F_\theta$ makes an acute angle with the tangent vector to ϕ. Therefore, the inward pointing perpendicular to VF_θ along ϕ is the one to the right of ∇F_θ: $(F_{\theta y}, -F_{\theta x})$. Now the derivative of $F_{\theta\theta}$ along $\Gamma(\theta)$, oriented inward from ϕ, must be decreasing. Thus $F_{\theta y}F_{\theta\theta x} - F_{\theta x}F_{\theta\theta y} < 0$.

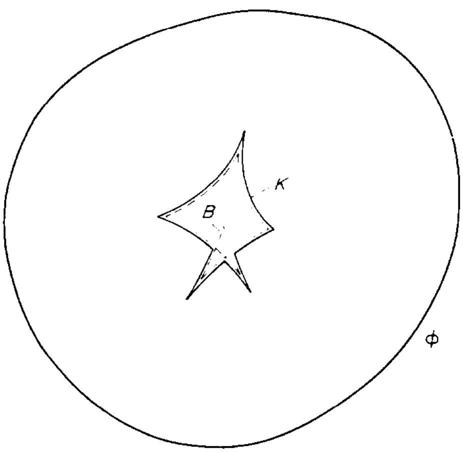

Fig. 3.

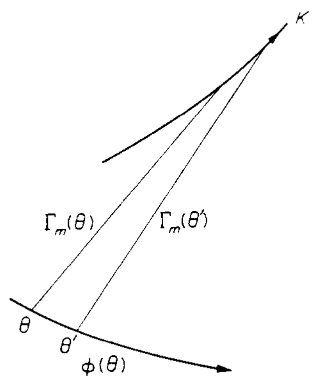

Fig. 4.

From this we conclude that the curvature of B, oriented by θ is positive. It remains for us to show that this orientation is the one for which K is traversed in a clockwise direction. The tangent vector to K has been computed to be

$$\frac{F_{\theta\theta\theta}}{F_{\theta x}F_{\theta\theta y} - F_{\theta y}F_{\theta\theta x}}(F_{\theta y}, -F_{\theta x}).$$

Therefore, the orientation of K agrees with that along the rays $\Gamma(\theta)$ when $F_{\theta\theta\theta} > 0$ and is opposite when $F_{\theta\theta\theta} < 0$. In the first case $F_{\theta\theta\theta} > 0$, increasing θ implies that the lengths of the rays $\Gamma_m(\theta)$ are increasing. Thus the region covered by the segments $\Gamma_m(\theta)$ lies to the left of $\Gamma(\theta)$ and the exterior of K lies to the right of K in this case; see Fig. 4. Since the curvature of K is positive, it is concave toward the exterior.

Similarly, if $F_{\theta\theta\theta} < 0$ and the orientation of K is opposite to that of $\Gamma(\theta)$, then the lengths of the rays of $\Gamma_m(\theta)$ are decreasing with increasing θ. Hence the interior of K lies to the left and the exterior to the right. Again we conclude that K is concave toward its exterior, proving the theorem.

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